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THE STOCHASTIC CASH BALANCE PROBLEM WITH AVERAGE COMPENSATING-BALANCE REQUIREMENTS* †

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We consider the problem of managing two assets, cash and an earning asset, when net cash flows are stochastic and when there are transfer costs for transferring assets from one form to the other. Previous work on the stochastic cash-balance problem has assumed holding costs for holding excess cash and penalty costs for holding insufficient cash, with these costs assessed *per period* (the same period in which there is a single decision or transfer opportunity and a single random cash flow). This formulation is appropriate when a firm faces *minimum* (or zero) compensating-balance requirements, but not when the compensating-balance requirement involves an *average* deposit balance over a number of decision periods. A dynamic programming model is presented which appropriately represents the relevant cost function for a firm facing an average compensating-balance requirement. The dynamic programming solution to a numerical example is compared to that of a static two-sided (s, S) policy; the optimal dynamic programming solution represents an 18% reduction in relevant costs in the example.

1. Introduction

The problem under consideration involves the management of cash and short-term financial assets for a firm facing a compensating-balance requirement specified as an *average* balance over a number of days (e.g., weekly, bi-weekly, or monthly). Daily net cash flows are partially[1] unpredictable and are treated as stochastic (specifically, as independent random variables). We consider only two assets: cash, and some interest-bearing asset.

At the end of a period, cash holdings in excess of the compensating-balance requirement incur an opportunity cost, in that they could have been invested in the interest-bearing asset. A cash level below the requirement presumably incurs some penalty cost, which will be assumed to be proportional to the shortfall. Transactions costs of converting excess cash into the earning asset and vice-versa make it uneconomic to "even up" daily, and create the management decision problem studied (in variations) here and in the references.

Previous Work

A number of researchers have analyzed various versions of the cash balance problem (see references). A review of that work will not be presented here, except to point out that almost all of the previous work has assumed implicitly or explicitly that there is either a zero compensating-balance requirement or a requirement of the *minimum* form. (References [7] and [11] do focus on the requirement to maintain *average*

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[1] The mean of the distribution of net cash flow may differ from day to day, reflecting the possibility of forecasting seasonal systematic effects. However, no forecast revision process of the type described in [10] is allowed for in this model.

compensating balances; unfortunately, they restrict their form of operating policy to a stationary type which is not optimal under the new setting.)

This paper will present a dynamic programming formulation of the stochastic cash balance problem under an average compensating-balance requirement. The problem is relevant because, as Gibson states, "··· balance requirements often apply to average, not daily, balances" [8, p. 387]. Other writers, including Frost [7], Miller and Orr [14], [18], and Stone [26] make the same point. §2 of the paper presents the dynamic programming model. §3 describes a realistic numerical example which has been solved by both the dynamic programming model and by a static two-sided (s, S) policy.

2. The Dynamic Programming Model

Time periods (days) are numbered backwards from some horizon N days away. Let X_n represent the opening cash balance on day n, *prior to* any transfer decision. Let Y_n represent the cash balance immediately *after* a transfer action (if any). We allow for both fixed and variable components of transfer costs $A(X_n, Y_n)$:

$$(1) \qquad A(X_n, Y_n) = \begin{cases} K + k(Y_n - X_n) & \text{if } Y_n > X_n, \\ 0 & \text{if } Y_n = X_n, \\ Q + q(X_n - Y_n) & \text{if } Y_n < X_n. \end{cases}$$

The average compensating balance requirement is given exogenously as R per day over an N-day averaging period, or NR dollar-days. The opportunity cost of holding a greater-than-R average balance is c_o per dollar per N-day period; we presume there is a per-dollar net[2] penalty cost c_u per N-day period for holding less than the "required" average R balance.

Random daily net cash inflows (+ or −) are denoted by ξ_n, independent random variables with known probability density function $p_n(\xi_n)$. These random flows occur after transfer action (if any) has been taken. Thus the equation relating successive daily cash balances for days n and $(n - 1)$ is:

$$(2) \qquad X_{n-1} = X_n + (Y_n - X_n) + \xi_n = Y_n + \xi_n .$$

Let S_n represent the cumulative sum of daily *closing* balances from the first day (N) through day $n + 1$ inclusive; then

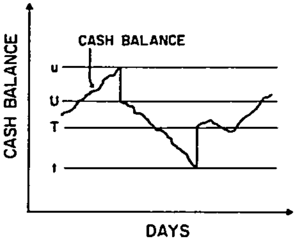
$$(3) \qquad S_n = \textstyle\sum_{i=n+1}^{N} (Y_i + \xi_i).$$

The state vector will be (S_n, X_n). The usual dynamic programming return function will be defined as $f_n(S_n, X_n)$ equal to the minimum expected cost from day n through day 1, given (S_n, X_n) and assuming optimal decisions are made from day n through day 1.

One-Period Problem

Now consider the last day (day 1). Action must be decided upon at the beginning of the day concerning a potential transfer; subsequently, random cash flow ξ_1 will occur; and, since this is the last day of the N-day averaging period, overage and underage costs will be assessed as appropriate. We may write the following dynamic pro-

[2] I.e., the gross penalty cost less c_o, the rate at which interest is earned on the earning asset.



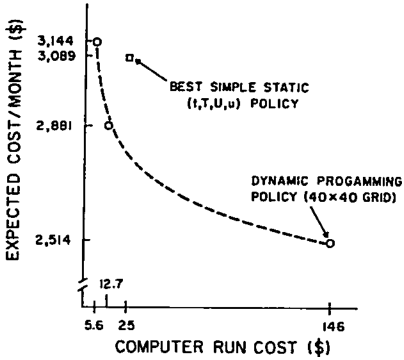


FIGURE 1. The Simple Static Policy (t, T, U, u).
FIGURE 2. Results of Dynamic Programming Model.

gramming return function for period 1:

$$f_1(S_1, X_1) = \text{Minimum}_{Y_1 \geq W}\Big\{A(X_1, Y_1)$$

$$(4) \qquad + c_u \int_{-\infty}^{NR-S_1-Y_1} [NR - S_1 - Y_1 - \xi_1] p_1(\xi_1)\, d\xi_1$$

$$+ c_o \int_{NR-S_1-Y_1}^{\infty} [\xi_1 + S_1 + Y_1 - NR] p_1(\xi_1)\, d\xi_1\Big\}$$

where W represents a lower bound ($W > 0$) on the ending balance, so as to keep the probability of a negative balance sufficiently low[3]. Denote the two integral terms in (4) by the symbol $L_1(Y_1 \mid S_1)$. Then the function $L_1(\cdot \mid \cdot)$ is convex in Y_1, and the required minimization of equation (4) is obtained [15, pp. 477–479] as follows:

$$(5) \qquad Y_1^* = \begin{cases} T_1, & X_1 \leq t_1 \\ X_1, & t_1 < X_1 < u_1 \\ U_1, & u_1 \leq X_1 \end{cases} \quad \text{with} \quad t_1 < T_1 < U_1 < u_1$$

(or if the lower bound W is active, then $Y_1^* = W$). The policy described by (5) has been called a "simple" policy, or a two-sided (s, S) policy [15, p. 473]; see Figure 1 for illustration of a corresponding static or stationary policy.

n-Period Problem

The general recursion relation corresponding to (4) for period n requires the expression of the state vector at period $n - 1$ in terms of the state vector and decision variable at period n. By definition,

$$S_{n-1} = S_n + Y_n + \xi_n$$

[3] Stone indicates that "... a firm can tolerate fairly large fluctuations in its cash balances as long as net collected balances are not negative" [26, p. 73].

and

$$X_{n-1} = Y_n + \xi_n .$$

Now (4) may be modified in the usual manner to represent the return function for day n as follows:

$$\text{(6)} \quad f_n(S_n, X_n) = \text{Min}_{Y_n \geq W}\Big\{A(X_n, Y_n) + \int_{\xi_n=-\infty}^{\infty} f_{n-1}(S_n + Y_n + \xi_n, Y_n + \xi_n)p_n(\xi_n)d\xi_n \quad \text{for } n = 2, 3, \cdots, N$$

(4) and (6) for $n = 1, 2, \cdots, N$ may be solved recursively to yield the optimal policy $Y_n^*(S_n, X_n)$ as a tabled function of the two-dimensional state vector at each decision stage.

Linking Successive Averaging Periods

It is important to note that solving (4) and (6) for $n = 1, \cdots, N$ represents the optimal solution to a single-averaging-period problem. In reality, the end of each averaging period is the beginning of the next. Thus, while the average balance earned in the previous period is not "carried over", there is a linking between successive periods through the actual level of ending cash balance, $X_0 = Y_1 + \xi_1$, which by definition becomes the opening balance X_N for the next averaging period.

A formal way to model this linking is to modify (4) to represent follow-on effects by adding an additional term as follows:

$$\begin{aligned} f_1(S_1, X_1) = \text{Minimum}_{Y_1 \geq W}\Big\{&A(X_1, Y_1) \\ \text{(7)} \qquad &+ c_u \int_{-\infty}^{NR-S_1-Y_1} [NR - S_1 - Y_1 - \xi_1]p_1(\xi_1)d\xi_1 \\ &+ c_o \int_{NR-S_1-Y_1}^{\infty} [\xi_1 + S_1 + Y_1 - NR]p_1(\xi_1)d\xi_1 \\ &+ \int_{-\infty}^{\infty} f_N(0, Y_1 + \xi_1)p_1(\xi_1)d\xi_1\Big\}. \end{aligned}$$

Then solution of (7) and (6) could proceed as follows:

1. Solve (4) for $f_1(\ )$.
2. Solve (6) for $f_2(\ )$, and recursively for $f_3(\ ), \cdots, f_N(\ )$.
3. Solve (7) for $f_1(\ )$, using $f_N(\ )$ just obtained.
4. Go to Step 2 and cycle between 2 and 3 until sufficient convergence occurs in the optimal decision tables $Y_n^*(S_n, X_n)$.

As a practical matter, the linking of successive averaging periods seems overly sophisticated, given the other assumptions of the model. The only effect such linking would have would be the effect of entering a new averaging period with a particular level of cash balances as opposed to some other level. Since the very problem we

TABLE 1

Data for Numerical Example

Transfer Costs	Fixed (dollars)	Variable ($ per thousand dollars transferred)
Cost of transfer from interest earning asset to cash	K = $20.00	$k = 0.5$
Cost of transfer from cash to interest earning asset	Q = $20.00	$q = 0.5$
Overage and underage costs (in $ per thousand of deviation from requirement)		
Cost per day of being above balance requirement: $c_o/N = 0.25$		
Cost per day of being below balance requirement: $c_u/N = 0.375$		
Time Horizon: N = 20 days (period over which the average balance is computed).		

are studying allows the decision-maker to effectively transfer dollar-days around within the averaging period, it is reasonable to assume that ignoring this minor linking will not penalize our decisions significantly. Therefore, for the remainder of the paper we focus solely on (4) and (6) as the optimizing model.

3. A Numerical Example

In this section, a numerical example is described which has been solved both by the dynamic programming model of §2 and by a simple static (t, T, U, u) policy for comparison purposes.

The numbers and parameter values selected for the example represent our estimates of a realistic problem. The distribution of daily net cash inflows was taken directly from research by Homonoff and Mullins [11] to be Normal[4] with mean of $4,000 and standard deviation of $580,000; this distribution and its parameters closely approximated the daily net cash inflows of an actual U.S. corporation over an 11-month period (see [11]). That corporation was required to maintain an average compensating balance of $3 million; we selected this identical requirement for our example. Other data required for the example is described in Table 1.

Dynamic Programming Model

The continuous state space (S_n, X_n) of the dynamic programming model of §2 was modified to a discrete two-dimensional grid for computational purposes. The accuracy of the discrete approximation to the underlying continuous process is dependent on the size of the grid and the number of points on it. A large grid with many points (i.e., small step size from one point to the next) will be more accurate than a smaller grid with fewer grid points, since interpolation between the two-dimensional grid points will increase in accuracy as the distance between the points decreases. However, the computer program to compute (4) and (6) will take increasingly larger amounts of time to run as the grid is made more dense, and a tradeoff between accuracy and computational expense must be made.

Table 2 presents three alternative grid sizes and corresponding dynamic programming results for our numerical example. Figure 2 illustrates the tradeoff between in-

[4] For computational purposes, the normal distribution was approximated by a discrete probability function.

TABLE 2

Results of Dynamic Programming Model

Cash Balances X_n			Cumulative Cash Balances S_n			Expected Cost ($ per month)	Cost of Computer Run ($)
X_{min} (thousands)	X_{max}	# of intervals	S_{min} (thousands)	S_{max}	# of intervals		
0	9000	10	0	95000	20	3144.9	5.60
0	9500	20	0	95000	20	2881.34	12.72
0	9750	40	0	97500	40	2514.42	146.24

creased grid size (and computation cost) and improvements in cost from increased accuracy.

Comparison Model

It is not a simple matter to find a reasonable policy against which we can compare the results of the dynamic programming calculations. Since the problem is clearly nonstationary, one might consider a dynamic version of the so-called "simple" policy. That is, a (t_n, T_n, U_n, u_n) policy for day n. However, derivation of optimal parameter values for such a policy is essentially as much computational effort as the complete dynamic programming solution to any given numerical problem. This did not seem a good use of computer time to us.

We finally settled on a comparison model using a *static* (stationary) policy of the simple (t, T, U, u) form. While this type of policy totally ignores the nonstationary aspects of the problem, it does provide a simple benchmark against which the improvement from dynamic programming can be assessed.

Even in the stationary case of the simple policy there is no analytical way to determine optimal parameter values, so a computer simulation was performed to obtain approximately-optimal parameter values. This simulation was used first in an exploratory manner with 25 simulations per policy to locate the region in parameter space which seemed appropriate for further study (see Table 3). Then a second series of runs were made with 100 simulations each to obtain a more precise estimate of both optimal parameter values and the resulting minimum average cost (see Table 4). The lowest[5] average monthly cost, $3089.40, was associated with a symmetric policy where $t = 2250$, $T = 2500$, $U = 3500$, $u = 3750$, (see last row in Table 4). Assuming the computer run costs for running the simulations of Table 4 (run time only) where $25, the best static policy is also plotted in Figure 2.

Conclusions

From Figure 2, the most accurate dynamic programming model produced the lowest expected costs ($2514.42). This cost is 18% less than the best static (t, T, U, u) policy, which cost $3089.40. It should be emphasized that once the optimal dynamic programming solution has been obtained, it is available in tabular form and can be

[5] It is likely that the parameter values producing the lowest average cost in Table 4 are not the "optimal" parameter values. However, given the standard deviation of the cost values (see Table 4), it is likely that the cost figure $3089.40, chosen as it was by *ranking* sample results, is an *underestimate* of the cost of the corresponding parameter values from the theory of order statistics. Thus the effect of possibly missing the optimal parameter values is at least partially offset by the order-statistics effect.

TABLE 3

Results from the First Set of Simulations
(25 simulations per policy tested)

Parameters (in thousands of dollars) t	T	U	u	Average Cost ($ per month)	Standard Deviation of Cost (in collars)
0	2000	4000	6000	7,413.7	4,616.4
1000	2000	4000	5000	5,200.5	2,641.4
2300	2600	3750	4050	3,820.9	1,169.4
2000	3000	3000	4000	3,804.8	1,351.7
2250	3000	3000	3750	3,745.7	770.5
1500	2500	3500	4500	3,722.5	2,059.7
2450	2800	3625	4025	3,653.6	1,352.9
2400	2800	3675	4025	3,611.0	1,561.6
2500	3000	3500	4000	3,431.7	1,445.7
2000	2750	3250	4000	3,401.8	1,647.1
2350	2600	3750	4050	3,365.5	1,796.0
1750	2500	3500	4250	3,362.7	2,071.3
2300	2600	3800	4050	3,319.8	1,440.6
2400	2800	3625	4075	3,246.8	1,162.6
2000	2500	3500	4000	3,244.9	1,917.0
2250	2500	3500	3750	3,175.0	1,296.0
2400	2850	3625	4025	3,150.0	1,364.8
2300	2600	3750	4160	3,101.6	1,681.5
2300	2650	3625	4050	3,075.5	1,393.6
2400	2800	3625	4025	3,073.5	1,325.1

used without further computer cost as long as the problem description remains essentially unchanged with respect to the model. Thus, even though computer programming (as opposed to the solution run) of the dynamic programming model was certainly more involved and more costly than that of the simulation, the initial programming represents a one-shot investment, and the initial major run (costing $146.20 in Figure 2) represents an investment which need not be repeated until elements of the problem change significantly.[6]

Further Research

We have totally ignored the question of maturity of the earning asset. Unless one is purchasing overnight Repurchase Agreements, then using our model one faces the problem of possibly selling an asset prior to its maturity, which is undesirable for a number of practical reasons. Direct inclusion of maturity life would quickly make the dynamic programming formulation computationally infeasible; some innovative way of incorporating maturities is needed to cope fully with this aspect of the problem.

Also, Homonoff and Mullins [11], in their study of actual daily net cash inflows, found that two distinct patterns were present in mean cash flows: a day-of-week pattern, and a separate pattern based on dividing the month into three 10-day periods (for details see [11]). The model presented here was analyzed using a data-generating process for daily net cash inflows which did not contain these time dependencies.

[6] If the dynamic programming model is modified to take into account forecasts of cash flows, and if these forecasts change over time, then the solution would have to be recomputed each time a forecast revision occurred. Nevertheless, no reprogramming would be required.

TABLE 4

Simulation of Static Simple (t, T, U, u) Policies
(100 *simulations per policy tested*)

Parameters (in thousands of dollars)				Average Cost ($ per month)	Standard Deviation of Cost (in dollars)
t	*T*	*U*	*u*		
2500	3000	3500	4000	3,703.1	1,230.9
2500	2750	3750	4000	3,648.4	1,372.6
2400	2800	3625	4075	3,614.8	1,461.4
2400	2800	2675	4025	3,632.7	1,348.5
2345	2790	3585	4005	3,605.9	1,816.5
2400	2800	3600	4000	3,586.7	1,470.1
2340	2816	3569	4007	3,578.0	1,486.8
2355	2790	3575	4005	3,575.3	1,654.1
2340	2816	3574	4002	3,559.6	1,330.0
2400	2850	3625	4025	3,406.9	1,385.3
2345	2816	3569	4002	3,405.2	1,309.3
2345	2790	3575	4015	3,397.8	1,608.4
2450	2800	3625	4025	3,389.8	1,330.1
2345	2790	3575	4005	3,352.4	1,531.1
2400	2800	3625	4025	3,250.9	1,399.6
2340	2816	3569	4002	3,294.3	1,408.5
2345	2816	3569	4002	3,291.3	1,442.1
2300	2800	3500	4000	3,262.5	1,398.5
2345	2800	3575	4005	3,199.0	1,404.9
2250	2500	3500	3750	3,089.4	1,359.7

While (4) and (6) could readily incorporate different mean cash flows based on the day of the month *n*, the question of a benchmark alternative policy is harder to answer in this more complex world. Finally, in [11] the corporate officer responsible for cash and earning asset transfers felt that major influences on cash flows were known to him through his general business experience. The current research has not shed light on the question of whether the latter factor can in fact dominate the actual results of a cash balance problem.

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